Mittagsseminar

Variable Metric Random Pursuit

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joint work with
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Abstract

A continuous function f is to be minimized. Similar as in the discrete setting, people often use Randomized Search Heuristics for this task. While they often give good results in practice, most of them lack a thorough theoretical convergence analysis.

In this talk, we present and analyze Variable Metric Random Pursuit. This is an iterative algorithm where in every step a new approximation is calculated by searching along a randomly chosen direction. However, typically not every direction yields the same progress. The algorithm tries to "learn" successful search directions and samples them more often.

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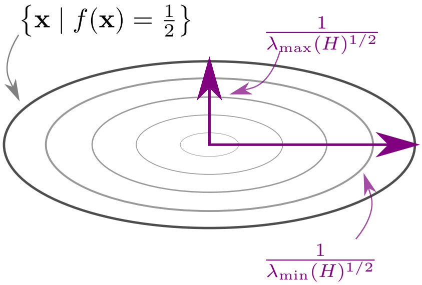
The recipe

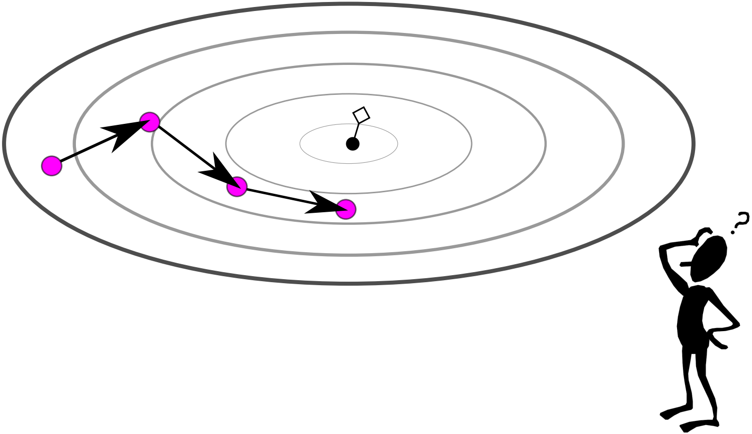
Problem: $\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$	$\rightarrow$	fancy heuristic	$\rightarrow$	apply		done ✓
$\downarrow$		$\downarrow$				
(general) problem class non-convex functions	$\rightarrow$	clean code, main features of state of the art algorithm	$\rightarrow$	analyze		PHD ✓
$\downarrow$		$\downarrow$				
convex functions this talk: quadratic	$\rightarrow$	2 features from 2 algorithms	$\rightarrow$	analyze		this works ✓

Quadratic functions

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T H \mathbf{x}$$



Where to go?

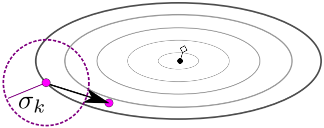


Where to go?

$$\mathbf{x}_{k+1} = \begin{cases} \mathbf{x}_k + \sigma_k \mathbf{u}_k & \text{if better} \\ \mathbf{x}_k & \text{otherwise} \end{cases}$$

(1+1)-Evolution Strategy (ES)

- $\mathbf{u}_k \sim \mathcal{N}(0, I_n)$
- σ_k "stepsize", empirically determined $\mathbf{Pr}\left[f(\mathbf{x}_k + \sigma_k \mathbf{u}_k) \leq f(\mathbf{x}_k)\right] = \frac{1}{5}$



$\downarrow$

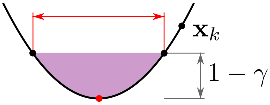
Random Pursuit

- $\mathbf{u}_k$ unifom random unit vector (direction)
- σ_k "best possible": line search in direction $\mathbf{u}_k$

Line Search

Sufficient decrease $0 < \gamma \leq 1$:

$$f(\mathbf{x}_{k+1}) \leq (1-\gamma) f(\mathbf{x}_k) + \gamma \min_{t \in \mathbb{R}} f(\mathbf{x}_k + t\mathbf{u}_k)$$



- We don't care about implementation: oracle
- count the number of oracle calls/line searches
- this talk: $\gamma = 1$
- (1+1)-ES with optimal scale σ_k:
  sample only one point $\Leftrightarrow$ γ is "large enough" *in expectation*

Convergence I

Let $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T H \mathbf{x}$.

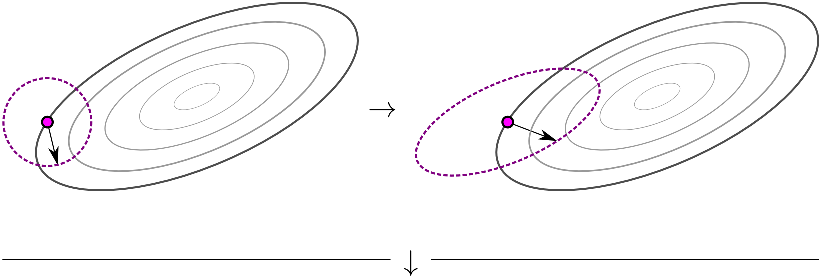
After k steps:

$$\mathbb{E}\left[f(\mathbf{x}_k)\right] \leq \left(1 - \frac{\lambda_{\min}(H)}{\mathrm{Tr}[H]}\right)^k f(\mathbf{x}_0)$$

- for $H = I_n$ the factor equals $\left(1 - \frac{1}{n}\right)$, this is optimal
- what if $H \neq I_n$?

Some directions are better than others. . .

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \sigma_k \mathbf{u}_k$$



Fixed Metric Random Pursuit

- $\mathbf{u}_k := \frac{\mathbf{v}_k}{\|\mathbf{v}_k\|_2}$ random unit vector (direction) $\mathbf{v}_k \sim \mathcal{N}(0, \Sigma)$
- σ_k "best possible": line search in direction $\mathbf{u}_k$

Convergence II

Let $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T H \mathbf{x}$.

After k steps:

$$\mathbb{E}\left[f(\mathbf{x}_k)\right] \leq \left(1 - \frac{\lambda_{\min}(H\Sigma)}{\mathrm{Tr}[H\Sigma] + 2\lambda_{\max}(H\Sigma)}\right)^k f(\mathbf{x}_0)$$

- for $\Sigma = H^{-1}$ the factor equals $\left(1 - \frac{1}{n+2}\right)$
- how can we find Σ?

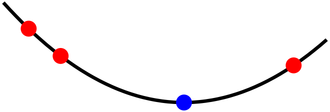
How to find Σ?

Goal: find $\Sigma^{-1} = H$

Solution: minimize $g(X) = \frac{1}{2} \|X - H\|_F^2$ **with Random Pursuit**

X_k is a $n \times n$ matrix:

- search direction $U_k = \mathbf{u}_k \mathbf{u}_k^T$, with $\mathbf{u}_k \sim \mathcal{N}(\mathbf{0}, I_n)$
- stepsize: exact line search, use the fact that g is a quadratic function: 3 function evaluations are sufficient



$$g(X_k + \alpha U_k) = \underbrace{\frac{1}{2} \|X_k - H\|_F^2 + \frac{1}{2} \|U_k\|_F^2}_{\text{constant}} + \alpha \langle X_k - H, U_k \rangle$$

Observation: $\langle H, U_k \rangle = f(\mathbf{x} + \mathbf{u}_k) - 2f(\mathbf{x}) + f(\mathbf{x} - \mathbf{u}_k)$

Randomized Hessian Update

Randomized Hessian Update [Leventhal, Levis, 2011]

- $U_k = \mathbf{u}_k \mathbf{u}_k^T$, with $\mathbf{u}_k \sim \mathcal{N}(\mathbf{0}, I_n)$ search direction
- $X_{k+1} = X_k - \langle X_k - H_k, U_k \rangle U_k$ (exact line search)

$$\mathbb{E}\left[\|X_k - H\|_F^2\right] \leq \left(1 - \frac{2}{n(n+2)}\right)^k \|X_0 - H\|_F^2$$

$\downarrow$

Variable Metric Random Pursuit

- Random Pursuit with $\mathbf{u}_k \sim \mathcal{N}(0, \Sigma_k)$
- update $\Sigma_k^{-1} \to \Sigma_{k+1}^{-1}$ with Randomized Hessian Update

Does this help?

- If the error $\|X_k - H\|_F$ is relatively large we don't know.
- If the error is relatively small $\|X_k - H\|_F \leq c\lambda_{\min}(H)$, $c < 1$:

$$1 - \frac{\lambda_{\min}(H\Sigma_k)}{\mathrm{Tr}[H\Sigma_k] + 2\lambda_{\max}(H\Sigma_k)} \leq 1 - \frac{(1-c)^2}{n+2}$$

- the progress is independent of H!

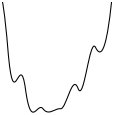
Outlook and Open problems

1. Dependence on $\lambda(H)$
2. When can we stop (with the learning)?

(general) problem class non-convex functions	$\rightarrow$	clean code, main features of state of the art algorithm	$\rightarrow$	analyze
$\downarrow\uparrow$		$\downarrow\uparrow$		
convex functions	$\rightarrow$	2 features from 2 algorithms	$\rightarrow$	analyze

3. Metric learning scheme from state of the art algorithm (update directly on Σ_k instead of Σ_k^{-1})
4. *some* non-convex functions

Thank you



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