ISMP 2018 - Bordeaux

Approximate Composite Minimization: Convergence Rates and Examples

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July 4, 2018

[AISTATS 2018] Adaptive balancing of gradient and update computation times

Convex Optimization

Black-Box Optimization

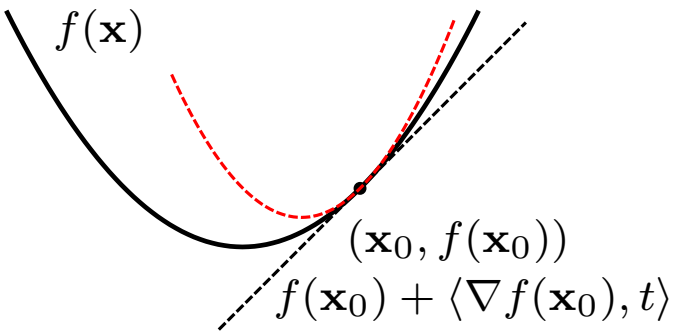
$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \qquad \mathbf{x} \rightarrow \boxed{\phantom{f(\mathbf{x})}} \rightarrow f(\mathbf{x})$$

- $f\colon \mathbb{R}^n \to \mathbb{R}$, assume $f \in C^1$
- oracle access to f $(f(\mathbf{x}), \nabla f(\mathbf{x}))$ or just $(\nabla_i f(\mathbf{x}))$
- Approximate solution $\mathbf{x}_\epsilon$: $f(\mathbf{x}_\epsilon) - \min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \leq \epsilon$
- $N(f, \mathcal{A}, \epsilon)$ number of oracle calls for algorithm $\mathcal{A}$ to find an approximate solution
- Worst case complexity of algorithm $\mathcal{A}$ on a class $\mathcal{F}$: $N(\mathcal{A}, \epsilon) = \max_{f\in\mathcal{F}} N(f, \mathcal{A}, \epsilon)$

Function Classes

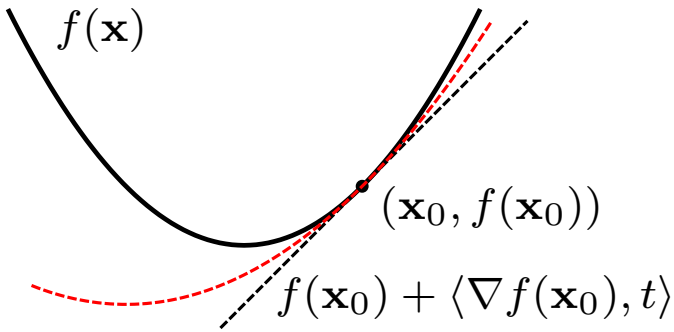
Upper bound: $f \in C^1_{\mathbf{L}}$

$$f(\mathbf{y}) \leq \underbrace{f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{1}{2} \|\mathbf{y} - \mathbf{x}\|^2_{\mathbf{L}}}_{=: u^{\mathbf{L}}_{\mathbf{x}}(\mathbf{y})}$$



Lower bound: $f \in C_\mu$

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{\mu}{2} \|\mathbf{y} - \mathbf{x}\|^2_{\mathbf{L}}$$



Consequences:

- $\mathbf{x}^\star := \operatorname{argmin}_{\mathbf{x}} f(\mathbf{x})$ unique
- $f(\mathbf{x}) - f(\mathbf{x}^\star) \leq \frac{1}{2\mu} \|\nabla f(\mathbf{x})\|^2_{\mathbf{L}^{-1}}$
- condition number $\kappa := \|\mathbf{L}\| / \mu$

$$\|\mathbf{x}\|_{\mathbf{L}} := \langle \mathbf{x}, \mathbf{L}\mathbf{x} \rangle$$

$$\mu\mathbf{L} \preceq \nabla^2 f(\mathbf{x}) \preceq \mathbf{L}$$

general framework

$$F(\mathbf{x}) := f(\mathbf{x}) + \Psi(\mathbf{x})$$

Setting:

- $f \in C^1_{\mathbf{L}}$, $f \in C^1_{\mu_f}$ for $\mu_f \geq 0$.
- $\Psi \in C_{\mu_\Psi}$ for $\mu_\Psi \geq 0$.

Goal:

$$\min_{\mathbf{x} \in \mathbb{R}^n} F(\mathbf{x}) := f(\mathbf{x}) + \Psi(\mathbf{x})$$

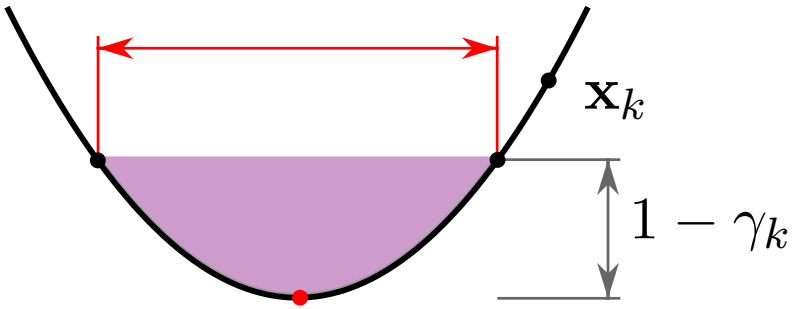
Method 0: $\mathbf{x}_0 \in \mathbb{R}^n$

$$\mathbf{x}^\star = \min_{\mathbf{u} \in \mathbb{R}^n} F(\mathbf{x}_0 + \mathbf{u})$$

This problem is too hard! Define easier subproblems!

Relaxing Accuracy:

Introduce relative accuracy parameter $\gamma_k \geq 0$:



Method 1: $\mathbf{x}_0 \in \mathbb{R}^n$

$$F(\mathbf{x}_{k+1}) \leq (1-\gamma_k)\underbrace{F(\mathbf{x}_k)}_{\text{old value}} + \gamma_k \underbrace{\min_{\mathbf{u}\in\mathbb{R}^n} F(\mathbf{x}_k+\mathbf{u})}_{\text{best value}}$$

Verifying the condition requires function evaluations, which might be elusive!

Option II: Optimize the upper bound

Upper bound:

$$F(\mathbf{x}_k + \mathbf{u}) \leq \underbrace{u^{\mathbf{L}}_{\mathbf{x}_k}(\mathbf{x}_k + \mathbf{u}) + \Psi(\mathbf{x}_k + \mathbf{u})}_{=:m^{\mathbf{L}}_{\mathbf{x}_k}(\mathbf{x}_k+\mathbf{u})}$$

Note: It suffices to pick $\mathbf{L}$ s.t. $F(\mathbf{x}_k + \mathbf{u}) \leq m^{\mathbf{L}}_{\mathbf{x}_k}(\mathbf{x}_k + \mathbf{u})$, i.e. $f(\mathbf{x}_k + \mathbf{u}) \leq u^{\mathbf{L}}_{\mathbf{x}_k}(\mathbf{x}_k + \mathbf{u})$ is not required.

Method 2: $\mathbf{x}_0 \in \mathbb{R}^n$

$$F(\mathbf{x}_{k+1}) \leq m^{\mathbf{L}}_{\mathbf{x}_k}(\mathbf{x}_{k+1}) \leq (1-\gamma_k)\underbrace{F(\mathbf{x}_k)}_{\text{old value}} + \gamma_k \underbrace{\min_{\mathbf{u}\in\mathbb{R}^n} m^{\mathbf{L}}_{\mathbf{x}_k}(\mathbf{x}_k + \mathbf{u})}_{\text{best value}}$$

This requires computation of $\nabla f(\mathbf{x}_k)$ (to evaluate $u^{\mathbf{L}}_{\mathbf{x}_k}(\mathbf{x}_k + \mathbf{u})$) and $\Psi(\mathbf{x}_k + \mathbf{u})$ instead of $F(\mathbf{x}_k + \mathbf{u})$.

Complexity Results

$$N(\textbf{Method 2}, \epsilon) = \min \left\{ \underbrace{\frac{2D^2}{\bar{\gamma}\epsilon}}_{\text{smooth case}}, \underbrace{\frac{1+\mu_\Psi}{(\mu_f+\mu_\Psi)\bar{\gamma}} \ln\left(\frac{F(\mathbf{x}_0)-F(\mathbf{x}^\star)}{\epsilon}\right)}_{\text{strongly convex case}} \right\}$$

$D = \max_{\mathbf{y}:\, F(\mathbf{y}) \leq F(\mathbf{x}_0)} \|\mathbf{y}-\mathbf{x}^\star\|_{\mathbf{L}}$, average accuracy $\bar{\gamma} = \frac{1}{k}\sum_{i=1}^k \gamma_k$.

- Both μ_f and μ_Ψ appear in the rate, thus $\mathbf{L}$ should ideally approximate the curvature of *both* f an Ψ.
- Extends and unifies **[Stich et al. 13], [Qu et al. 15], [Tappenden et al. 16], [Mutny, Richtárik, 18], [...]**

Examples

Gradient Method

$$\mathbf{L} = L\mathbf{I}_n$$

$$\mathbf{x}_{k+1} = \operatorname*{argmin}_{\mathbf{u}\in\mathbb{R}^n} \left\{ f(\mathbf{x}_k) + \langle \nabla f(\mathbf{x}_k), \mathbf{u} - \mathbf{x}_k \rangle + \frac{L}{2} \left\| \mathbf{u} - \mathbf{x}_k \right\|^2 + \Psi(\mathbf{x}_k + \mathbf{u}) \right\}$$

$\mathbf{L}^{-1}$ hard to compute: use an (arbitrary) iterative method to solve

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k) + \gamma_k \operatorname*{argmin}_{\mathbf{u}\in\mathbb{R}^n} \left\{ \langle \nabla f(\mathbf{x}_k), \mathbf{u} - \mathbf{x}_k \rangle + \frac{1}{2} \left\| \mathbf{u} - \mathbf{x}_k \right\|_{\mathbf{L}}^2 + \Psi(\mathbf{x}_k + \mathbf{u}) \right\}$$

Requires full $\nabla f(\mathbf{x}_k)$, which might be elusive for $n \gg 1$.

Random Methods

$$T(\text{iteration}) \ll T(\nabla f)$$

Random Pursuit

Choose a (random) sketch matrix $\mathbf{U}_k \in \mathbb{R}^{n\times m}$, $\operatorname{rank}\mathbf{U}_k = m$.

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k) + \gamma_k \underbrace{\operatorname*{argmin}_{\mathbf{u}\in span\,\mathbf{U}_k} \left\{ \langle \nabla f(\mathbf{x}_k), \mathbf{u}\rangle + \frac{1}{2}\|\mathbf{u}\|^2_{\mathbf{L}} + \Psi(\mathbf{x}_k+\mathbf{u}) \right\}}_{=:\ \operatorname*{argmin}_{\mathbf{u}\in R^n} m^{\mathbf{L}}_{\mathbf{x}_k|\mathbf{U}_k}(\mathbf{x}_k+\mathbf{u})}$$

Requires computation of $\mathbf{U}_k^\top \nabla f(\mathbf{x}_k)$.

Example: (Block)-Coordinate Descent

- $\mathbf{U}_k^\top \mathbf{U}_k = \mathbf{I}_m$, $\mathbf{U}_k^\top \nabla f(\mathbf{x}_k) = [\nabla_{i_1} f(\mathbf{x}_k), \dots, \nabla_{i_m} f(\mathbf{x}_k)]^\top$.
- Assume Ψ (block)-separable structure:
  $\Psi(\mathbf{x}_k + \mathbf{U}_k\mathbf{z}) = \Psi_{|\mathbf{U}_k^\perp}(\mathbf{x}_k) + \Psi_{|\mathbf{U}_k}(\mathbf{x}_k + \mathbf{U}_k\mathbf{z})$

How does this fit in the framework?

Option I: The γ_k-approximate solution on $m^{\mathbf{L}}_{\mathbf{x}_k|\mathbf{U}_k}$ can be seen as a γ'_k-approximate solution on $m^{\mathbf{L}}_{\mathbf{x}_k}$, $\gamma'_k \leq \gamma_k$.

Option II: For certain distributions of $\mathbf{U}_k$ the results can be explicitly stated.
Example:
If $\mathbb{E}\,\mathbf{U}_k\mathbf{U}_k^\top = \frac{m}{n}\mathbf{I}_n$ the complexity increases by a factor of $\frac{n}{m}$.

[KSJ18] also states explicit results for parallel updates.

Application: (Block) Coordinate Descent

$$[\nabla_{i_1} f(\mathbf{x}), \ldots, \nabla_{i_m} f(\mathbf{x})]$$

Block Coordinate Descent: sketch matrix $\mathbf{U}_k$, $\mathbf{U}_k^\top \mathbf{U}_k = \mathbf{I}_m$

m ($\ll n$)-dimensional subproblem:

$$m^{\mathbf{L}}_{\mathbf{x}_k}(\mathbf{x}_{k+1}) \leq (1-\gamma_k) F(\mathbf{x}_k) + \gamma_k \min_{\mathbf{u} \in span\, \mathbf{U_k}} m^{\mathbf{L}}_{\mathbf{x}_k}(\mathbf{x}_k + \mathbf{u})$$

Iteration comprises two steps:

1. Compute $\mathbf{U}_k^\top \nabla f(\mathbf{x}_k) = [\nabla_{i_1} f(\mathbf{x}_k), \dots \nabla_{i_m} f(\mathbf{x}_k)]^\top$
2. Compute approx. minimizer $\mathbf{x}_{k+1}$

Two different costs! Examples:

- Cache misses, expensive operations to compute $\nabla_i f(\mathbf{x}_k)$
- Hard subproblem, $\Psi(\mathbf{x}) = \|\mathbf{x}\|_{TV}$

Towards optimal balancing

Goal: Set accuracy γ_k to optimally 'balance' the costs

Feasible strategy:

- Suppose we use an interative algorithm to approximately solve the model $m^{\mathbf{L}}_{\mathbf{x}_k}$. I.e. $\mathbf{x}_k = \mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_t, \ldots, \mathbf{y}_T = \mathbf{x}_{k+1}$
- Each intermediate solution $\mathbf{y}_t$ corresponds to an approximate solution with parameter γ_k^t:

$$m^{\mathbf{L}}_{\mathbf{x}_k}(\mathbf{y}_t) \leq (1-\gamma_k^t)F(\mathbf{x}_k) + \gamma_k^t \min_{\mathbf{u} \in span\, \mathbf{U_k}} m^{\mathbf{L}}_{\mathbf{x}_k}(\mathbf{x}_k + \mathbf{u})$$

- Thus we can express the progress in terms of iterations on the model:

$$p_k(t) := \underbrace{\gamma_k^t}_{\text{relative progress}} \underbrace{\left(F(\mathbf{x}_k) - \min_{\mathbf{u} \in span\, \mathbf{U_k}} m^{\mathbf{L}}_{\mathbf{x}_k}(\mathbf{x}_k + \mathbf{u}) \right)}_{\text{scale}}$$

Optimal balancing

- Each (inner) iteration $(\mathbf{y}_t \to \mathbf{y}_{t+1})$ denotes one 'unit' of time.
- Let $c_k = T(\mathbf{U_k}^\top \nabla f(\mathbf{x}_k))$ denote the time to compute the required coordinates of the gradient.

Optimal number of inner iterations $t^\star$:

$$t_k^\star := \operatorname*{argmax}_t \underbrace{\frac{p_k(t)}{t + c_k}}_{\text{progress per time spent}}$$

Can we compute $t_k^\star$? Can we ~~compute~~ approximate $t_k^\star$?
Predict t_k based on observations at iteration $k-1$. **Goal:** $t_k \approx t_k^\star$

(Constant) strategies for approximating $t_k^\star$

Constant Strategies:

one: Set $t_k \equiv 1$, $\forall k \geq 0$. — arbitrary constant

comp: $t_k = c_k$, 'balance' *time* — "best practice"

Can we measure $p_k(t_k)$? $\quad p_k(t_k) = F(\mathbf{x}_k) - m_{\mathbf{x}_k}^{\mathbf{L}}(\mathbf{x}_{k+1})$

And $p_k'(t_k)$?

$$\frac{d}{dt}\underbrace{\frac{p_k(t_k)}{t + c_k}}_{\text{progress per time spent}} = \frac{p_k'(t_k)(t_k + c_k) - p_k(t_k)}{(t_k + c_k)^2} =: g_k(t_k)$$

Can be estimated using $p_k'(t_k) \approx p_k(t_k) - p_k(t_{k-1})$.

(Adaptive) strategies for approximating $t_k^\star$

General adaptive strategy: Initialize $t_0 := 1$

Each iteration:

1. Estimate $g_k(t_k)$
2. $t_{k+1} = \mathcal{A}(t_k, g_k)$

Example strategy: **add**

$$t_{k+1} = \begin{cases} t_k + 1 & \text{if } g_k(t_k) > 0 \\ \max\{t_k - 1, 1\} & \text{otherwise} \end{cases}$$

Two more adaptive strategies:

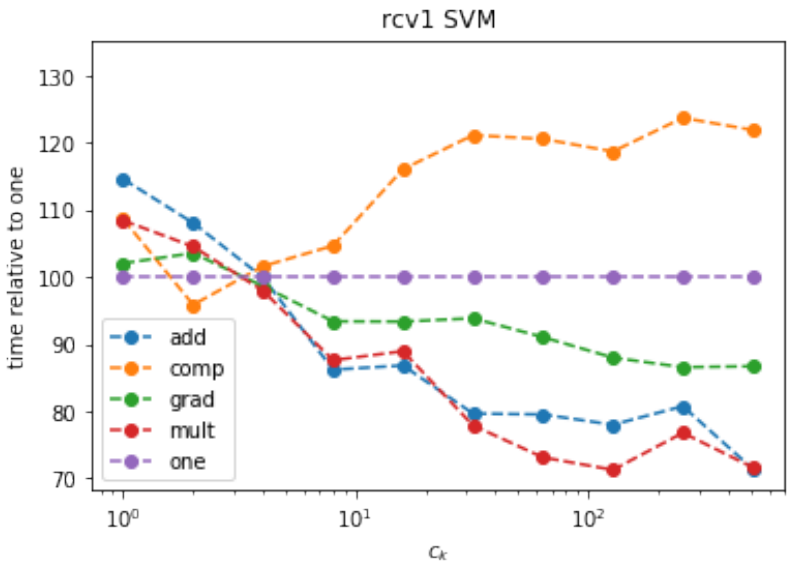
mult: $t_{k+1} = \{2t_k, \text{if } g_k(t_k) > 0, \max\{t_k - 1, 1\}, \text{else}$

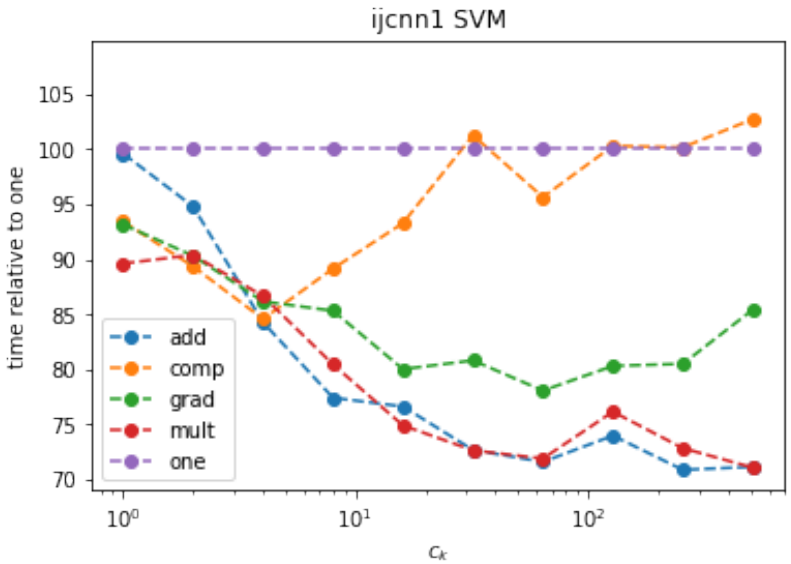
grad: $t_{k+1} = t_k + g_t$

Experiments

Improvement in Time

Time relative to constant strategy one:

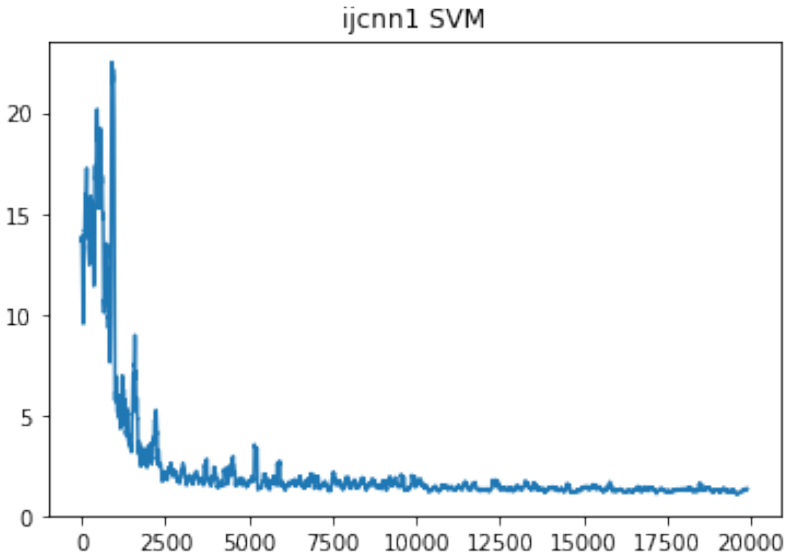




- **adaptivity** is important! precise rule less important
- "best practice" **comp** is not performing well!

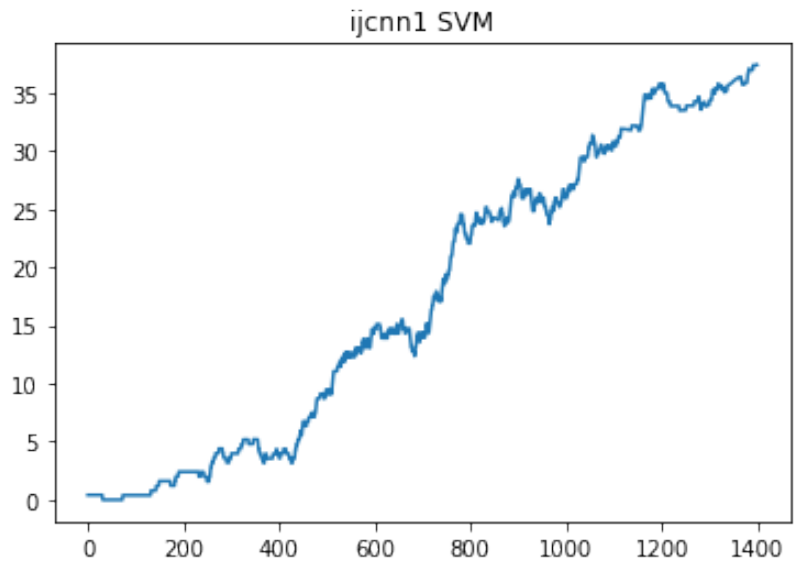
A closer look

mult strategy:



t_k values v.s. iteration k

Problem difficulty: (simulated)



t_k required to reach accuracy $\gamma_k \equiv 0.1$ vs. iteration k

less time is spent as the subproblems get harder

Conclusion

Contributions & Open Problem

Theory: We present a sound theoretical framework

- where subproblems need to be solved just approximately (with arbitrary iterative solver)
- convergence rate depends on average quality of approximation
- parallel, distributed and primal-dual extensions

Practice: We observe

- **adaptivity** is important, not all subproblems are the same!

Open problem: Proof for adaptive schemes missing!

- The proofs do not extend to the fully-adaptive setting, i.e. when γ_k depends on $\mathbf{x}_{k+1}$ (as it is the case for the adaptive strategies).

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